

ON LEVI'S THEOREM FOR LEIBNIZ ALGEBRAS

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ABSTRACT. A Lie algebra over a field of characteristic 0 splits over its soluble radical and all complements are conjugate. I show that the splitting theorem extends to Leibniz algebras but that the conjugacy theorem does not.

Let L be a finite-dimensional left Leibniz algebra over a field of characteristic 0. I denote by d_a the left multiplication operator $d_a: L \rightarrow L$ defined by $d_a(x) = ax$ for all $a, x \in L$.

I call the subspace $\langle x^2 \mid x \in L \rangle$ spanned by the squares of elements of L the Leibniz kernel of L and denote it $\text{Leib}(L)$. It is an abelian ideal of L , $L/\text{Leib}(L)$ is a Lie algebra and $\text{Leib}(L)L = 0$. Let $R = R(L)$ be the soluble radical of L . Then $\text{Leib}(L) \subseteq R$.

Theorem 1. *There exists a semi-simple subalgebra S of L such that $S + R = L$ and $S \cap R = 0$.*

Proof. Put $K = \text{Leib}(L)$. By Levi's Theorem (see Jacobson [1, Chapter III, p. 91]), there exists a semi-simple subalgebra S^*/K of L/K such that $S^* + R = L$ and $S^* \cap R = K$. It is sufficient to prove that S^* splits over K , so we may suppose $R = K$.

Since $KL = 0$, L may be considered as a left module for the semi-simple Lie algebra L/K . By Whitehead's Theorem (see Jacobson [1, Chapter III, Theorem 8, p.79]), this module is completely reducible. Thus there exists a submodule S complementing K . Since $LS \subseteq S$, we have $SS \subseteq S$ and S is a subalgebra. \square

Example 2. Let S be a simple Lie algebra and let K be isomorphic to S as left S -module. I denote by x' the element of K corresponding to $x \in S$ under this isomorphism. I make K into a Leibniz module by defining the right action to be 0. Let L be the split extension of K by S . Then L is a Leibniz algebra and $\text{Leib}(L) = K$. The space $S_1 = \{(s, s') \mid s \in S\}$ is another subalgebra complementing K since, using the module isomorphism, we have

$$(s, s')(t, t') = (st, st') = (st, (st)').$$

For any $x = (s, k) \in L$, the inner derivation $d_x = d_s$, $d_x(S) \subseteq S$ and so also $\exp(d_x)(S) \subseteq S$. Thus S and S_1 are not conjugate.

REFERENCES

1. N. Jacobson, *Lie algebras* Interscience, New York-London (1962).

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